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Random covering of the circle: the configuration-space of the free deposition process

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Abstract

Consider a circle of circumference 1. Throw at random n points, sequentially, on this circle and append clockwise an arc (or rod) of length s to each such point. The resulting random set (the free gas of rods) is a collection of a random number of clusters with random sizes. It models a free deposition process on a 1D substrate. For such processes, we shall consider the occurrence times (number of rods) and probabilities, as n grows, of the following configurations: those avoiding rod overlap (the hard-rod gas), those for which the largest gap is smaller than rod length s (the packing gas), those (parking configurations) for which hard rod and packing constraints are both fulfilled and covering configurations. Special attention is paid to the statistical properties of each such (rare) configuration in the asymptotic density domain when $ns = \rho$, for some finite density ρ of points. Using results from spacings in the random division of the circle, explicit large deviation rate functions can be computed in each case from state equations. Lastly, a process consisting in selecting at random one of these specific equilibrium configurations (called the observable) can be modelled. When particularized to the parking model, this system produces parking configurations differently from Rényi's random sequential adsorption model.

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1. Introduction

Random sequential adsorption (RSA) models have attracted the attention of many researchers, starting with the pioneer work on the parking problem of Rényi [19] (see [5] and the references therein for a survey in physics). In this model, cars (or rods) are thrown sequentially at random on a street (an interval); a new impinging car leaves without parking if some overlap with previously parked cars occurs and parks otherwise. This out of equilibrium process continues until space-filling of the street is attained (a packing property), ending in a parking configuration. As underlined by Widom in 1973 (see [5], p 1312), there is a neat distinction between this sequential space-filling problem and analogous equilibrium problems based on general configuration-space considerations. Here it would rather be asked for the occurrence probability of a parking configuration when cars are thrown sequentially on the line but without rejection, in a free deposition process. In this work, we shall exclusively focus on this equilibrium version of the Rényi problem. Parking configurations are those for which parked cars do not overlap (a hard rod property) and which saturate space (a packing property). We shall therefore also study separately the occurrence times and probabilities of two types of configurations in a free deposition process: those avoiding rod overlap (the hard-rod gas) and those for which the largest gap is smaller than the rod length (the 'packing gas'). Occurrence probabilities of covering configurations are also studied for their connection with the continuum one-dimensional percolation model [3]. Without any loss of generality, we shall work on the circle rather than on the interval. Let us discuss our motivations in more detail.

What is Rényi's parking construction? It is an algorithm which sorts out parking configurations sequentially by forcing one of them to occur. Its telling feature lies in the fact that the underlying mechanism mimics a natural phenomenon: rods of size s (small) impinge at random on a substrate (say the unit circle instead of a street) in a random deposition process; they do not stick to the substrate unless their whole surface adheres completely to it. Passing through intermediate hard rod states, the algorithm naturally ends up in a parking configuration with a random number $\mathcal{N}_{\text{park}}(s)$ of rods. This random set has prescribed statistical properties which have been extensively studied; for example, its limiting average space-filling rate is Rényi's universal jamming constant $s\mathbf{E}\mathcal{N}_{\text{park}}(s) \to_{s\downarrow 0} 0.748\ldots$(see [10, 17] for much more).

In a free deposition process producing a free gas of rods, rods impinge sequentially at random on a substrate and always stick to the substrate at the impact site; so, here, Nature acts freely without constraint. One can think of many natural phenomena where this picture is relevant. Proceeding in this way, we shall show that only very few (exceptional) configurations are of the parking type; furthermore, they are observable in the density domain only, when the number of points n satisfies $ns = \rho$ with $\rho \in \left(\frac{1}{2}, 1\right)$, in the thermodynamic limit. Now assume that for physical reasons (such as evidence of interactions between rods), only these specific configurations are worthy of interest in a free deposition process. Then, the statistical question consisting in selecting one of them at random naturally arises; the output of this random choice will be called the 'observable'. The random pick algorithm which we shall introduce will mimic the random observation process of one of these (happy few) parking configurations: it consists in selecting one out of the total ensemble of parking configurations (with enlarged probability space) while giving equal weight to each of them. If $\mathcal{N}_{\text{park}}(s)$ is the (random) number of rods of the observable, we shall see that now $s\mathcal{N}_{\text{park}}(s) \to_{s\downarrow 0} \log 2$ almost surely (a.s.). In this alternative approach to *Rényi*'s producing parking configurations, the random subset of the circle arising from random throws of s-rods on it is considered globally: the whole equilibrium configuration is rejected if it is not a parking configuration. When no further information on the observable is available, the chosen uniform prior distribution on parking configurations seems to be well-defined.

Parking configurations are of interest in a free deposition process but they are not the only ones; hard rod configurations (for their connection with simple fluid models), packing and covering configurations are also remarkable. Their occurrence times and probabilities, as n varies, are thus studied as a function of rod length s going to 0, to complete the picture; it is shown that, although the range of their occurrence is now quite large, they all occur with exponentially small probability in the density domain when $ns = \rho$ for some finite density $\rho > 0$. Explicit large deviation rate functions and state equations are supplied in each specific case. The question of the one which can typically be observed in this region is also considered; the construction is similar to that for parking configurations. Some statistical properties of the randomly chosen set (the observable) can thus be obtained similarly.

2. Circle covering problems: the free gas of rods

In this section, we introduce the random set under study throughout. This allows us to give a definition of the free deposition process.

2.1. Random division of the circle: basic facts

Let us first recall some well-known facts on the random division of the circle into n arcs [7, 11, 12, 20]).

Consider a circle of unit circumference, say $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Throw at random $n > 1$ points $X_1, \ldots, X_n$ on this circle (thus, with $X_1, \ldots, X_n$ independent and identically distributed, say iid, and uniform). Start with some randomly chosen point out of the n points; call it $X_{(1),n}$. Next, consider the ranked set of points $(X_{(m),n}, m = 1, \ldots, n)$, obtained from $X_1, \ldots, X_n$ while turning clockwise on the circle, starting from $X_{(1),n}$. Let $S_{m,n} = l(X_{(m+1),n}, X_{(m),n})$, $m = 1, \ldots, n-1$ be the consecutive spacings (i.e. the arc lengths between consecutive points), with $S_{n,n} := l(X_{(1),n}, X_{(n),n})$ closing the loop. The random vector of spacings $(S_{m,n}; m = 1, \ldots, n)$ has singular uniform Dirichlet(n) density function on the simplex [18]

$$f_{S_{1,n},\ldots,S_{n,n}}(s_1, \ldots, s_n) = (n-1)! \cdot \delta_{(\sum_{m=1}^{n} s_m - 1)}. \tag{1}$$

As a result, $S_{m,n} \stackrel{d}{=} S_n, m = 1, \ldots, n$, independently of m and the individual spacings are all identically distributed ($\stackrel{d}{=}$). Their common distribution on the interval (0, 1) is given by $\mathbf{P}(S_n > s) = (1-s)^{n-1}$, with mean value $\mathbf{E}S_n = 1/n$.

Consider next the sequence $(S_{(m),n}; m = 1, \ldots, n)$ obtained while ranking the spacing vector $(S_{m,n}; m = 1, \ldots, n)$ according to ascending size, hence with $S_{(1),n} < \cdots < S_{(n),n}$. The $S_{(m),n}$ distribution has been known since [22], with some preliminary work on the subject done by W A Whitworth (1897) and R A Fisher (1929) (see e.g. [3, 9] for historical background). In particular, with x_+ the positive part of x

$$\mathbf{P}(S_{(1),n} > s) = (1-ns)_+^{n-1} \qquad \text{and} \qquad \mathbf{P}(S_{(n),n} \leqslant s) = \sum_{p=0}^{n} (-1)^p \binom{n}{p} (1-ps)_+^{n-1}$$

are the smallest and largest spacing distributions. From this, one can prove that, with E an exponentially distributed random variable with mean 1

$$n^2 S_{(1),n} \stackrel{d}{\to} E \qquad \text{and} \qquad \frac{n}{\log n} S_{(n),n} \stackrel{a.s.}{\to} 1 \qquad \text{as} \quad n \uparrow \infty.$$

In a random division of the circle, although the consecutive spacings are identically distributed, the smallest spacing is of order n^{-2} while the largest is of order $\frac{1}{n}\log n$.

In the problems considered here, the joint law of the smallest and largest spacings will clearly be involved. This problem has pre-occupied statisticians for a while, starting with P Lévy, D Darling and L Weiss. We recall here the main results. Following Darling (see [1] and the references therein), it can be proved in different ways that

$$\mathbf{P}(S_{(1),n} > s_1, S_{(n),n} \leqslant s_2) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (1 - (ns_1 + m(s_2 - s_1)))_+^{n-1}. \tag{2}$$

Putting $s_1 = s, s_2 = 2s$ gives the probability of a s-parking configuration with n points. From (2), it can also be shown that as $n \uparrow \infty$

$$\left(n^2 S_{(1),n}, n\left(S_{(n),n} - \frac{1}{n}\log n\right)\right) \xrightarrow{d} (E, G) \tag{3}$$

where E and G are independent random variables with $\mathbf{P}(G \leqslant t) = \exp(-\exp(-t)), t \in \mathbb{R}$ a Gumbel distribution function.

2.2. The free gas of rods model

Let now $\mathcal{X}_n := \{X_1, \ldots, X_n\}$ be the above set of points drawn at random on the circle. Fix $s \in (0, 1)$ (as a fraction of the unit circle circumference, s is dimensionally the inverse of length). Consider the coarse-grained random set of arcs

$$\mathcal{X}_n(s) := \{X_1 + x, \ldots, X_n + x, 0 \leqslant x < s\} \tag{4}$$

appending clockwise an arc of 'length' s to each starting point of $\mathcal{X}_n$. For obvious reasons, we may call $\mathcal{X}_n(s)$ a free gas of rods. For these free gases, the following random variables are worthy of interest: the first is $P_n(s)$, which is the number of connected components of $\mathcal{X}_n(s)$ (which is also the number of its gaps). By convention, we set $P_n(s) = 0$ as soon as the circle is covered by $\mathcal{X}_n(s)$ (which is the event $S_{(n),n} \leqslant s$). The second is $\mathcal{L}_n(s)$, which is the total length of $\mathcal{X}_n(s)$. As there are $n - P_n(s)$ spacings covered by s and $P_n(s)$ gaps each contributing s to the covered length, it can be expressed as a contribution of two terms

$$\mathcal{L}_n(s) := \sum_{m=1}^{n} \min(S_{m,n}, s) = \sum_{m=1}^{n-P_n(s)} S_{(m),n} + s P_n(s). \tag{5}$$

The problems of determining the distributions of $P_n(s)$ and $\mathcal{L}_n(s)$ are solved for each fixed n and s. Asymptotic results are available as n goes to ∞ jointly with s going to 0, in particular in the case $ns = \rho \in (0, \infty)$ where ρ is the free gas density [12]. (Note that ns is the expected value of the number of arcs containing point x on the circle, which is independent of x from rotation invariance of the model.)

Finally, we shall need to introduce the following definitions. Let $\mathbb{X} := \{X_1, \ldots, X_n, \ldots\}$ be the set of a countable number of points drawn at random on the circle. Fix $s \in (0, 1)$. Consider the coarse-grained random set of arcs

$$\mathbb{X}(s) := \{X_1 + x, \ldots, X_n + x, \ldots, 0 \leqslant x < s\}$$

appending clockwise an arc of length s to each starting point of $\mathbb{X}$. Then $\mathbb{X}(s) = \cup_{n>1}\mathcal{X}_n(s)$, where $\mathcal{X}_{n+1}(s) = \mathcal{X}_n(s) \cup \{X_{n+1} + x, 0 \leqslant x < s\}$. The restriction of $\mathbb{X}(s)$ to the first n points of $\mathbb{X}$ is $\mathcal{X}_n(s)$, since inserting X_{n+1} within $\mathcal{X}_n$ (with Dirichlet(n) distribution for spacings) leads to the set $\mathcal{X}_{n+1}$ still with Dirichlet($n+1$) distribution for its spacings. We shall call $\mathcal{X}_n(s), n > 1$ the free deposition process. The probability attached to $\mathcal{X}_n \in \mathbb{T}^n$ is denoted by $\mathbf{P}_n \equiv \mathbf{P}$ while $\mathbb{X} \in \mathbb{T}^\infty$ is endowed with probability $\mathbb{P}$.

3. Covering configurations

Consider the random set $\mathbb{X}(s)$. We shall call a configuration of $\mathcal{X}_n(s)$ for which $S_{(n),n} \leqslant s$, a s-covering configuration with n points. The largest spacing between points being smaller than the rod length s, the circle is completely covered ($\mathcal{L}_n(s) > 1$). What then is the number of points for which a covering configuration is likely to occur in $\mathbb{X}(s)$?

Let us start then with the first occurrence time of a covering configuration: the cover probability $\mathbf{P}(P_n(s) = 0)$ is also the probability that the number of arcs of length s, say $N_{\text{cov}}(s)$, required to cover the circle is less than or equal to n. In other words, with

$$N_{\text{cov}}(s) := \inf(n : S_{(n),n} \leqslant s)$$

we have: $\mathbb{P}(N_{\text{cov}}(s) \leqslant n) = \mathbf{P}(S_{(n),n} \leqslant s)$. As a result, with $L(.)$ the slowly varying function at ∞ defined by $L(x) := \log(x \log x)$, $x > 1$, it holds

$$s\left\{N_{\text{cov}}(s) - \frac{1}{s}L(1/s)\right\} \xrightarrow{d}_{s\downarrow 0} G \tag{6}$$

where G has the Gumbel distribution [6, 8]. So, $\mathbb{E}N_{\text{cov}}(s) = \sum_{n>1} \mathbf{P}(S_{(n),n} > s) = \frac{1}{s}\{L(1/s) + \gamma + o(1)\}(s \downarrow 0)$, where $\gamma = \mathbf{E}(G)$ is Euler's constant [21].

Thus, a covering configuration is likely to occur when the number of points is of order $\frac{1}{s}L(1/s)$ although the range of n for which $\mathbf{P}(S_{(n),n} \leqslant s) > 0$ is $\{[1/s] + 1, \dots, \infty\}$, where $[x]$ is the integral part of x. Note also that if the number of points is of order $\frac{a}{s}L(1/s)$, for some $a > 0$, then, if $a > 1$, the circle is covered and the length of the largest cluster is 1, whereas if $a < 1$ the length of the largest cluster is 0: in this sense, we have a sol–gel phase transition at $a = 1$. At the critical point $a = 1$, if the number of points is of order $\frac{1}{s}\{L(1/s) + t\}$, $t \in \mathbb{R}$, the cover probability is $\mathrm{e}^{-\mathrm{e}^{-t}}$. Given the circle is not covered, there are $P(\mathrm{e}^{-t}) + 1$ macroscopic clusters where $P(\mathrm{e}^{-t})$ is a Poisson random variable with parameter e^{-t} [13]; the covered length tends to 1^- almost surely. This problem was, in part, considered in the statistical physics' literature in the context of the 1D continuum percolation model [3].

This allows us to obtain some insight into the covering configurations in the density domain. Indeed, when the number of points is of order $1/s$, there are too few points for a covering configuration to occur. One expects the probability of covering configurations to tend to zero exponentially fast. With $\rho \in (1, \infty)$, we can indeed prove (proceeding as in [4], see also subsection 7.1)

$$-\frac{1}{n}\log \mathbf{P}(nS_{(n),n} \leqslant \rho) \to F_{\text{cov}}(\rho) := 1 - \frac{p}{\rho} + \log\frac{p}{\rho} - \log(1 - \mathrm{e}^{-p}) \tag{7}$$

where thermodynamical 'pressure' $p \in \mathbb{R}$ and density $\rho \in (1, \infty)$ are related through the 'state equation'

$$\frac{1}{\rho} = \frac{1}{p} - \frac{\mathrm{e}^{-p}}{1 - \mathrm{e}^{-p}}. \tag{8}$$

In this sense, covering configurations are exceptional in the density domain in the free deposition process.

4. Hard-rod, packing, parking configurations: setting of the problem

Let us here introduce some definitions and preliminaries on the configurations we shall be concerned with in what follows.

Fix an arc length $s \in (0, 1)$. Throw independently n points uniformly on the circle and consider the set $\mathcal{X}_n(s)$, appending an arc of length s to each point. Suppose $S_{(1),n} > s$. Then, the number of $\mathcal{X}_n(s)$'s connected components, $P_n(s)$, is maximal ($P_n(s) = n$) and there is no overlap between the arcs: we obtain a monodisperse hard-rod gas with n points and size-s rods. Suppose in addition that the largest gap length, which is $S_{(n),n} - s$, is less than s. Then, no additional arc of length s can be added, anywhere on the circle, without provoking overlap which is a packing property. If both hard rod and packing conditions are fulfilled, we shall speak of a parking gas of rods. For any of these configurations (hard rod and parking), the length of the covered set is always $\mathcal{L}_n(s) = ns := \rho$. In the hard-rod case, values of ρ close to 0 (respectively 1) correspond to a dilute (dense) hard-rod gas. In the parking case, the range of ρ is $\left(\frac{1}{2}, 1\right)$. In sharp contrast with the free gas of rods model, these two models exhibit interactions as some constraints are imposed on the mutual rod positions.

We call a configuration of $\mathcal{X}_n(s)$ for which $S_{(1),n} > s$, a s-hard rod configuration: the number of connected components is maximal (there is no arc overlap). We call a configuration of $\mathcal{X}_n(s)$ for which $S_{(n),n} \leqslant 2s$, a s-packing configuration: the largest gap being smaller than the rod length s, there is no room to insert an additional rod without intersecting $\mathcal{X}_n(s)$. We call a configuration of $\mathcal{X}_n(s)$ for which $S_{(1),n} > s$ and $S_{(n),n} \leqslant 2s$, an s-parking configuration; as a subclass of hard rod configurations, the number of connected components is maximal (there is no arc overlap); due to lack of space, no additional arc of length s can be added, anywhere on the circle, without provoking overlap.

As we shall see below, we shall also need to introduce the following related class of configurations: the covering configurations of $\mathcal{X}_n(s)$, as those for which $S_{(n),n} \leqslant s$ constitute a subclass of packing configurations for which $S_{(n),n} \leqslant 2s$. We shall call configurations for which $s < S_{(n),n} \leqslant 2s$ strictly packing configurations: strictly packing configurations are thus packing configurations with gaps remaining to be filled.

5. Hard-rod configurations

In this section, we shall mainly be concerned with the occurrence probability of hard rod configurations. Let us start with the first occurrence time of some overlap.

A s-hard rod configuration of $\mathcal{X}_n(s)$ (i.e. with n points) occurs with probability $\mathbf{P}(S_{(1),n} > s)$, so

$$N_{\text{hrod}}(s) := \inf(n : S_{(1),n} \leqslant s)$$

is the first occurrence time of some overlap and $\mathcal{X}_n(s)$ is a hard-rod configuration if $n \leqslant N_{\text{hrod}}(s)$. Stated differently, suppose that if any two points of $\mathcal{X}_n$ are too close (say at distance less than s for some s), then the circle breaks, i.e. becomes topologically an interval: then $N_{\text{hrod}}(s)$ is the number of points needed to break the circle. The quantity $s - S_{1:N_{\text{hrod}}(s)}$ is the amplitude of the first overlap provoking breakage. From the definition of $N_{\text{hrod}}(s)$, we get: $\mathbb{P}(N_{\text{hrod}}(s) > n) = \mathbf{P}(S_{(1),n} > s)$ and so

$$\sqrt{s}N_{\text{hrod}}(s) \xrightarrow{d}_{s\downarrow 0} W_2$$

where W_2 is a Weibull random variable for which $\mathbf{P}(W_2 > t) = \exp(-t^2), t > 0$. Indeed $\mathbb{P}(N_{\text{hrod}}(s) > n) = \mathbf{P}(S_{(1),n} > st) = (1 - ns)_+^{n-1}$ and, as s is small [14]

$$\mathbb{P}(\sqrt{s}N_{\text{hrod}}(s) > t) \approx (1 - t\sqrt{s})^{t/\sqrt{s}} \to \mathrm{e}^{-t^2}. \tag{9}$$

Consequently, $\mathbb{E}N_{\text{hrod}}(s) = \sum_{n\geqslant 2} \mathbf{P}(S_{(1),n} > s)$, with $\mathbb{E}N_{\text{hrod}}(s) \sim_{s\downarrow 0} \frac{1}{2}\sqrt{\frac{\pi}{s}}$. Some overlap is therefore very likely to occur when the number of points is close to $s^{-1/2}$. We note however that the range of n for which $\mathbf{P}(S_{(1),n} > s) > 0$ is $\{2, \ldots, [1/s]\}$.

Let us now consider hard-rod configurations in the density domain: when the number of points is of order $1/s$ (the case with a density $\rho := ns$ of points), there are too many points for a non-overlapping configuration to occur. One expects that the probability of non-overlapping (hard-rod) configurations tends to zero exponentially fast. Proceeding as in (7), (8), we now find

$$-\frac{1}{n}\log \mathbf{P}(nS_{(1),n} > \rho) \to F_{\text{hrod}}(\rho) = 1 + p - \frac{p}{\rho} + \log\frac{p}{\rho} \tag{10}$$

with $\rho \in (0, 1)$. Here, thermodynamical 'pressure' $p > 0$ and density $\rho \in (0, 1)$ are related through the well-known Mayer 'state equation' for a hard-rod Tonks gas [15, 16]

$$\frac{1}{\rho} = \frac{1}{p} + 1. \tag{11}$$

The limiting rate function is explicit there with $F_{\text{hrod}}(\rho) = -\log(1-\rho)$. In the free deposition process, hard-rod configurations are exceptional in the density domain as well.

5.1. Random pick from hard-rod configurations

As mentioned in the introduction, we would like to model the observation process consisting in selecting one of the hard-rod configurations at random. The above considerations and facts suggest that the following construction could be of some relevance.

A s-hard rod configuration of $\mathcal{X}_n(s)$ (i.e. with n points) occurs with probability $\mathbf{P}(S_{(1),n} > s)$. Recall that the values of n for which the probability $\mathbf{P}(S_{(1),n} > s) > 0$ vary in the range $\{n_-(s) := 2, \ldots, n_+(s) := [1/s]\}$. As a result, with $\mathbf{I}(.)$ the set-indicator function, the quantity

$$\mu_{\text{hrod}}(s) := \mathbb{E}\sum_{n=2}^{n_+(s)} \mathbf{I}(S_{(1),n} > s) = \sum_{n=2}^{n_+(s)} \mathbf{P}(S_{(1),n} > s) \tag{12}$$

is the expected number of s-hard-rod configurations in the random set $\mathbb{X}(s) = \cup_{n>1}\mathcal{X}_n(s)$. This suggests introducing the following random observable: consider the set of all s-hard-rod configurations in $\mathbb{X}(s)$. Pick at random one of them and call it the 'observable'. Let $\mathcal{N}_{\text{hrod}}(s)$ be the number of rods of the output. Then, the law of $\mathcal{N}_{\text{hrod}}(s)$ will be given by

$$\mathbb{P}(\mathcal{N}_{\text{hrod}}(s) = n) = \mathbf{P}(S_{(1),n} > s)/\mu_{\text{hrod}}(s) \tag{13}$$

with $n \in \{2, \ldots, n_+(s)\}$. Note that the random variable $\mathcal{N}_{\text{hrod}}(s)$ is also characterized by

$$\mathbb{P}(\mathcal{N}_{\text{hrod}}(s) = n) = \frac{\mathbb{P}(N_{\text{hrod}}(s) > n)}{\mathbb{E}N_{\text{hrod}}(s)}.$$

Under this form, the distribution of $\mathcal{N}_{\text{hrod}}(s)$ is well-defined; it can be interpreted as the limiting forward recurrence time of some pure renewal process on the set $\mathbb{N}$ when discrete times separating consecutive arrivals in the queue are distributed like $N_{\text{hrod}}(s)$.

Assume $n \to \infty$, $s \to 0$ with $ns = \rho \in (0, 1)$. To have an estimate of the normalizing mean μ_{hrod} in (13), as a function of large n, for all values of s, we have to integrate the above probability (10), (11) over ρ in the corresponding range. We find the saddle-point estimate

$$\left\{\int_0^1 \mathrm{e}^{-nF_{\text{hrod}}(\rho)}\,\mathrm{d}\rho\right\}^{1/n} \to 1, n \uparrow \infty.$$

From this and (13), (10), we get $s\mathcal{N}_{\text{hrod}}(s) \to_{\mathbb{P}-a.s.} 0$ with large deviation rate function

$$s\log\mathbb{P}(s\mathcal{N}_{\text{hrod}}(s) = \rho) \to r_{\text{hrod}}(\rho) = \rho(F_{\text{hrod}}(0) - F_{\text{hrod}}(\rho)) = \rho\log(1-\rho). \tag{14}$$

A hard-rod gas is in one of the hard-rod states. If some observer picks at random one of these configurations, the observed limiting proportion of the circle which is covered is thus 0: the randomly chosen set looks void. This is not so paradoxical as it first may appear. Recall indeed that in the density domain, i.e. as $n \uparrow \infty$ and $s \downarrow 0$ while $ns = \rho \in (0, \infty)$, with $\theta := \mathrm{e}^{-\rho}$, the free gas of rods behaves as follows:

$$\frac{1}{\sqrt{n}}\{P_n(s) - n\theta\} \underset{n\uparrow\infty}{\overset{d}{\to}} \text{Gauss}(0, \sigma^2 = \theta(1-\theta))$$

$$\sqrt{n}\{\mathcal{L}_n(s) - (1-\theta)\} \overset{d}{\to} \text{Gauss}(0, \sigma^2 = \theta(1-\theta)).$$

In this Gaussian asymptotic regime, the physical image is as follows: the number of connected components diverges since $\frac{1}{n}P_n(s) \to \theta < 1$ (almost surely), thus with some overlap between rods; each connected component length has asymptotically equal distribution whose average size goes to zero and the expected covered length $\mathbf{E}\mathcal{L}_n(s)$ tends to $1-\theta$ which is a fraction of the circle length that does not go to 0 nor to 1 [12, 13]. Zooming on hard-rod configurations of this free gas (i.e. conditioning), θ is forced to 1 and so the fraction of the circle which is covered goes to 0. So, the fact that $\mathcal{N}_{\mathrm{hrod}}(s) \to_{a.s.} 0$ is not very informative, nor surprising. The information on the randomly chosen set rather seems to be encoded in the large deviation rate function (14).

6. Packing configurations

In this section, we shall mainly be concerned with the occurrence probability of packing configurations. The random pick algorithm to produce the observable is applied to strictly packing configurations.

6.1. First occurrence time of a packing configuration

Let us first investigate the following question: what is the number of points needed to reach a packing configuration? A s-packing configuration of $\mathcal{X}_n(s)$ (i.e. with n points) occurs with probability $\mathbf{P}(S_{(n),n} \leqslant 2s)$, so

$$N_{\mathrm{pack}}(s) := \inf(n : S_{(n),n} \leqslant 2s)$$

can be interpreted as the first time a packing configuration occurs in $\mathbb{X}(s)$. From this, we get $\mathbb{P}(N_{\mathrm{pack}}(s) \leqslant n) = \mathbf{P}(S_{(n),n} \leqslant 2s)$ and so, with $L(.)$ the slowly varying function at ∞ defined above, proceeding as for $N_{\mathrm{cov}}(s)$, we get

$$2s\left\{N_{\mathrm{pack}}(s) - \frac{1}{2s}L\left(\frac{1}{2s}\right)\right\} \xrightarrow{d}_{s\downarrow 0} G \tag{15}$$

where G has the Gumbel distribution. In particular, $\mathbb{E}N_{\mathrm{pack}}(s) := \sum_{n>1}\mathbf{P}(S_{(n),n} > 2s)$ satisfies $\mathbb{E}N_{\mathrm{pack}}(s) = \frac{1}{2s}\left\{L\left(\frac{1}{2s}\right) + \gamma + o(1)\right\}$ $(s \downarrow 0)$. Thus, as the number of points is of order $1/\sqrt{s}$ a first overlap is very likely to occur. Packing configurations are reached much later, when n is of the order $\frac{1}{2s}L\left(\frac{1}{2s}\right)$, just before covering when n is of the order $\frac{1}{s}L\left(\frac{1}{s}\right)$.

When the number of points is of order $1/s$ (the case with a density), there are too few points for a packing configuration to occur. One expects the probability of packing configurations to tend to zero exponentially fast. With $\rho \in (1/2, \infty)$, we can prove similarly

$$-\frac{1}{n}\log\mathbf{P}(nS_{(n),n} \leqslant 2\rho) \to F_{\mathrm{pack}}(\rho) := 1 - \frac{p}{\rho} + \log\frac{p}{\rho} - \log(1-\mathrm{e}^{-2p}) \tag{16}$$

where pressure $p \in \mathbb{R}$ and density $\rho \in (1/2, \infty)$ are now related through

$$\frac{1}{\rho} = \frac{1}{p} - \frac{2\,\mathrm{e}^{-2p}}{1-\mathrm{e}^{-2p}}. \tag{17}$$

6.2. Random pick out of strictly packing configurations

A strictly packing configuration of $\mathcal{X}_n(s)$ occurs with probability

$$\mathbf{P}(s < S_{(n),n} \leqslant 2s) = \mathbf{P}(S_{(n),n} \leqslant 2s) - \mathbf{P}(S_{(n),n} \leqslant s).$$

The values of n for which the probability $\mathbf{P}(s < S_{(n),n} \leqslant 2s) > 0$ vary in the range $\{n_-(s) := [1/(2s)] + 1, \ldots, n_+(s) := \infty\}$ and the quantity

$$\mu_{\text{pack}}(s) := \mathbb{E} \sum_{n=n_-(s)}^{\infty} \mathbf{I}(s < S_{(n),n} \leqslant 2s) = \sum_{n=n_-(s)}^{\infty} \mathbf{P}(s < S_{(n),n} \leqslant 2s) \tag{18}$$

is the expected number of strictly s-packing configurations in the random set $\mathbb{X}(s) = \cup_{n>1} \mathcal{X}_n(s)$. Note that $\sum_{n=n_-(s)}^{\infty} \mathbf{I}(s < S_{(n),n} \leqslant 2s) < \infty$, which is not the case for $\sum_{n=n_-(s)}^{\infty} \mathbf{I}(S_{(n),n} \leqslant 2s)$ and so the construction that follows would have failed if blindly applied to all packing configurations.

Picking at random a strictly s-packing configuration in $\mathbb{X}(s)$, we can define the random observable $\mathcal{N}_{\text{pack}}(s)$ as the cardinality of the random output of the trial. Its distribution is given by

$$\mathbb{P}(\mathcal{N}_{\text{pack}}(s) = n) = \frac{\mathbf{P}(s < S_{(n),n} \leqslant 2s)}{\mu_{\text{pack}}(s)} \qquad n \in \{[1/(2s)] + 1, \infty\}. \tag{19}$$

This is also

$$\mathbb{P}(\mathcal{N}_{\text{pack}}(s) = n) = \frac{\mathbb{P}(N_{\text{cov}}(s) > n) - \mathbb{P}(N_{\text{pack}}(s) > n)}{\mathbb{E}(N_{\text{cov}}(s)) - \mathbb{E}(N_{\text{pack}}(s))}$$

showing that the random variable $\mathcal{N}_{\text{pack}}(s)$ is well-defined.

Assume $n \to \infty$, $s \to 0$ with $ns = \rho$ for some density ρ. We would like to understand the distribution of $\mathcal{N}_{\text{pack}}(s)$ in this asymptotic regime. Recalling the asymptotic regime described in (7), (8) and (16), (17), with $\rho \in \left(\frac{1}{2}, 1\right)$ we now find

$$-\frac{1}{n} \log \mathbf{P}(\rho < nS_{(n),n} \leqslant 2\rho) \to F_{\text{pack}}(\rho) := 1 - \frac{p}{\rho} + \log \frac{p}{\rho} - \log(1 - \mathrm{e}^{-2p}). \tag{20}$$

The limiting large deviation rate function is thus the restriction of $F_{\text{pack}}(\rho)$ defined in (16), (17) on the interval $\rho \in \left(\frac{1}{2}, 1\right)$. On this interval, this function is minimum at $\rho = 1$ (pressure $p = 0$) and $F_{\text{pack}}(1) = \log(e/2)$. To have an estimate of the normalizing mean μ_{pack} in (19), as a function of (large) n, for all values of s, we have to integrate the above probability (20) over ρ in the corresponding range. We find the saddle-point estimate

$$\left\{ \int_{1/2}^{1} \mathrm{e}^{-nF_{\text{pack}}(\rho)}\, \mathrm{d}\rho \right\}^{1/n} \to_{n\uparrow\infty} 2/e$$

because the function $\mathrm{e}^{-F_{\text{pack}}(\rho)} > 0$ is maximal at $\rho = 1$, with value $\mathrm{e}^{-F_{\text{pack}}(1)} = 2/e$ there. This shows, from (19), (20), that, as s tends to 0, the random variable $s\mathcal{N}_{\text{pack}}(s)$ has a density $\mathbb{P}(s\mathcal{N}_{\text{pack}}(s) = \rho)$ at point ρ satisfying

$$\mathbb{P}(s\mathcal{N}_{\text{pack}}(s) = \rho) \sim_{s\downarrow 0} \{\mathrm{e}^{F_{\text{pack}}(1) - F_{\text{pack}}(\rho)}\}^{\rho/s}.$$

This density is very peaked around the value $\rho = 1$. As a result $s\mathcal{N}_{\text{pack}}(s) \overset{\mathbb{P}-a.s.}{\to}_{s\downarrow 0} 1$ and

$$\lim_{s\downarrow 0} s \log \mathbb{P}(s\mathcal{N}_{\text{pack}}(s) = \rho)) = r_{\text{pack}}(\rho) \tag{21}$$

with concave large deviation rate function

$$r_{\text{pack}}(\rho) = \rho(F_{\text{pack}}(1) - F_{\text{pack}}(\rho)) \leqslant 0 \qquad \rho \in \left(\tfrac{1}{2}, 1\right).$$

In this random selection process of a strictly packing configuration, the limiting proportion of the covered space is 1 and the randomly chosen set looks full. This sounds natural for (strictly) packing configurations.

7. Parking configurations

In this section, we study the occurrence probability and time of parking configurations: considering $N_{\text{hrod}}(s)$ and $N_{\text{pack}}(s)$ jointly, we get

$$\left(\sqrt{s}N_{\text{hrod}}(s), 2s\left\{N_{\text{pack}}(s) - \frac{1}{2s}L\left(\frac{1}{2s}\right)\right\}\right) \xrightarrow{d}_{s\downarrow 0} (W_2, G)$$

where W_2 and G are independent. As a whole, it is therefore very improbable that there is a finite $N_{\text{park}}(s)$ defined by

$$N_{\text{park}}(s) = \inf(n : S_{(1),n} > s, S_{(n),n} \leqslant 2s)$$

i.e. by the number of points needed to attain a parking configuration. Indeed, for realizations of $(X_1, \ldots, X_n, \ldots)$ for which some overlap occurs before a packing configuration is reached, $N_{\text{park}}(s)$ should be rejected at ∞ as a parking configuration will be observed with probability going to 0. Thus, there is no way to centre and scale $N_{\text{park}}(s)$ to make it converge in law and the question of the first occurrence time of some parking configuration is ill-defined (degenerate).

7.1. Parking configurations in the density domain

For the parking model, one expects that the probability of a parking configuration at density ρ (which is $\mathbf{P}(nS_{(1),n} > \rho, nS_{(n),n} \leqslant 2\rho)$) tends to 0 exponentially fast with n. This is because when n is of the order $1/s$, there are at the same time too many points for a hard-rod configuration to occur and too few points for a packing configuration to occur. We reproduce here a slight modification of a result given in [4] where the pair correlation function of parking configurations was computed and analysed in some detail. If we define the partition function in the parking case as

$$Z_n(\rho) := \int_{\mathbb{T}^n} \delta_{(\sum_{m=1}^n s_m - 1)} \prod_{m=1}^n \mathbf{I}\left(\frac{\rho}{n} < s_m \leqslant 2\frac{\rho}{n}\right) \mathrm{d}s_m \tag{22}$$

we have $\mathbf{P}(nS_{(1),n} > \rho, nS_{(n),n} \leqslant 2\rho) = (n-1)! \cdot Z_n(\rho)$, using (1). Now, with p the pressure, we shall rather consider the associated isobaric partition function

$$\begin{aligned}\tilde{Z}_n(p,\rho) &:= \int_{\mathbb{T}^n} \exp\left(-\frac{p}{\rho}n\sum_{m=1}^n\left(s_m - \frac{1}{n}\right)\right)\prod_{m=1}^n \mathbf{I}\left(\frac{\rho}{n} < s_m \leqslant 2\frac{\rho}{n}\right)\mathrm{d}s_m \\ &= \left\{\int_{\mathbb{T}} \exp\left(-\frac{p}{\rho}n\left(s_1 - \frac{1}{n}\right)\right)\mathbf{I}\left(\frac{\rho}{n} < s_1 \leqslant 2\frac{\rho}{n}\right)\mathrm{d}s_1\right\}^n \\ &= \left\{\int_\rho^{2\rho} \frac{1}{n}\exp\left(-\frac{p}{\rho}(u-1)\right)\mathrm{d}u\right\}^n\end{aligned}$$

omitting the Dirac delta constraint in (22) but coding it in the exponential prefactor. Using Stirling's formula, we get

$$\begin{aligned}(n-1)! \cdot \tilde{Z}_n(p,\rho) &\sim \left\{\frac{1}{\mathrm{e}}\int_\rho^{2\rho} \exp\left(-\frac{p}{\rho}(u-1)\right)\mathrm{d}u\right\}^n \\ &= \left\{\frac{\rho}{p}\exp\left(-\left(1-\frac{p}{\rho}\right)\right)(\mathrm{e}^{-p} - \mathrm{e}^{-2p})\right\}^n =: \exp(-nF_\rho(p)).\end{aligned}$$

Suppose $p = p_\rho$ satisfies condition $\partial_p F_\rho(p_\rho) = 0$. Define then $\tilde{Z}_n(\rho) := \tilde{Z}_n(p,\rho)$ and $F_{\text{park}}(\rho) := F_\rho(p)$ at $p = p_\rho$. In this way, we find

$$F_{\text{park}}(\rho) = 1 - \frac{p}{\rho} + \log\frac{p}{\rho} - \log(\mathrm{e}^{-p} - \mathrm{e}^{-2p}) \tag{23}$$

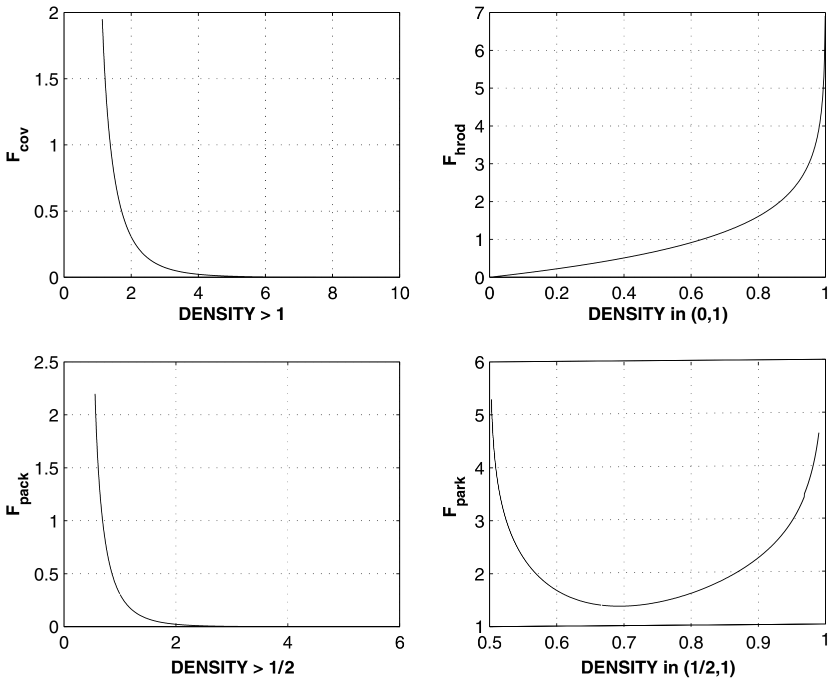

Figure 1. Top left: covering configuration, top right: hard rod, bottom left: packing, bottom right: parking configurations.

where simple computations show that $p \in \mathbb{R}$ and $\rho \in (1/2, 1)$ are related through the state equation

$$\frac{1}{\rho} = \frac{1}{p} + \frac{1 - 2\,\mathrm{e}^{-p}}{1 - \mathrm{e}^{-p}}. \tag{24}$$

Then, the condition expressed in the Dirac delta constraint in (22) does not change the expected value in $\tilde{Z}_n(\rho)$ but only suppresses the fluctuations of the underlying random walk variable (the equivalence of ensembles principle). These fluctuations follow the central limit theorem, so that in the thermodynamic limit, one could prove rigorously following [2] that $-\frac{1}{n}\log Z_n(\rho) \sim -\frac{1}{n}\log \tilde{Z}_n(\rho)$. In our context, this means

$$\lim_{n\uparrow\infty} -\frac{1}{n}\log \mathbf{P}(nS_{(1),n} > \rho, nS_{(n),n} \leqslant 2\rho) = F_{\mathrm{park}}(\rho). \tag{25}$$

The rate function $F_{\mathrm{park}}(\rho)$ is convex, minimal at $\rho = \log 2$ with $F_{\mathrm{park}}(\log 2) = 2\log 2$ and the parking probability $\mathbf{P}(nS_{(1),n} > \rho, nS_{(n),n} \leqslant 2\rho)$ tends to 0 exponentially fast according to (25). In a free deposition model, parking configurations are concentrated in the density domain and are exceptional there.

Note that results displayed in (7), (8), (10), (11) and (16), (17) can be obtained similarly, adapting the constraints on the spacings to each specific case. The expressions of the large deviation functions (7), (8), (16), (17) and (23), (24) seem to be novel.

We plot in figure 1 the functions F_{cov}, F_{hrod}, F_{pack} and F_{park} as a function of density in each specific case, using the equations of state. The Helmholtz free energy functions per particle f are related to the F through $f(\rho) = F(\rho) + \log\rho - 1$ and $f(\rho) = -\frac{p}{\rho} + g(\rho)$ where g is the Gibbs potential per particle.

7.2. Random pick from parking configurations

An s-parking configuration of $\mathcal{X}_n(s)$ (i.e. with n points) occurs with probability $\mathbf{P}(S_{(1),n} > s, S_{(n),n} \leqslant 2s)$ given in (2). The range of n for which the probability $\mathbf{P}(S_{(1),n} > s, S_{(n),n} \leqslant 2s) > 0$ is $\{n_-(s) := [1/(2s)] + 1, \ldots, n_+(s) := [1/s]\}$, thus in the density domain with $\rho \in (1/2, 1)$. Let

$$\mu_{\text{park}}(s) := \sum_{n=n_-(s)}^{n_+(s)} \mathbf{P}(S_{(1),n} > s, S_{(n),n} \leqslant 2s) \tag{26}$$

be the expected number of s-parking configurations in $\mathbb{X}(s)$.

Pick at random a s-parking configuration in $\mathbb{X}(s)$ and let $\mathcal{N}_{\text{park}}(s)$ be the number of rods of the output. From this definition, we get

$$\mathbb{P}(\mathcal{N}_{\text{park}}(s) = n) = \mathbf{P}(S_{(1),n} > s, S_{(n),n} \leqslant 2s)/\mu_{\text{park}}(s) \tag{27}$$

with $n \in \{n_-(s), \ldots, n_+(s)\}$.

Suppose $n \to \infty$ together with $s \to 0$, with $ns = \rho \in (1/2, 1)$. To have an estimate of the normalizing mean μ_{park} in (27), as a function of (large) n, for all values of s, we have to integrate the parking probability over ρ in the corresponding range. From (25), we find the saddle-point estimate

$$\left\{\int_{1/2}^{1} \exp(-nF_{\text{park}}(\rho))\,\mathrm{d}\rho\right\}^{1/n} \to_{n\uparrow\infty} \sup_{\rho\in(1/2,1)} \exp(-F_{\text{park}}(\rho)) = 1/4$$

because the function $\exp(-F_{\text{park}}(\rho)) > 0$ is maximal at $\rho = \log 2$, with value $1/4$ there. This shows, from (25), (27), that, as s tends to 0, $s\mathcal{N}_{\text{park}}(s)$ has a density at ρ verifying

$$\mathbb{P}(s\mathcal{N}_{\text{park}}(s) = \rho) \sim_{s\downarrow 0} \{4\exp(-F_{\text{park}}(\rho))\}^{\rho/s}$$

concentrating on $\rho = \log 2$. As a result $s\mathcal{N}_{\text{park}}(s) \overset{\mathbb{P}-a.s.}{\to}_{s\downarrow 0} \log 2 \approx 0.693\ldots$ and

$$\lim_{s\downarrow 0} s \log \mathbb{P}(s\mathcal{N}_{\text{park}}(s) = \rho)) = r_{\text{park}}(\rho) \tag{28}$$

with large deviation rate function $r_{\text{park}}(\rho) = \rho(F_{\text{park}}(\log 2) - F_{\text{park}}(\rho)) \leqslant 0$. In this random selection procedure of the observable parking configuration, the limiting fraction of the covered space is $\log 2$ which is slightly less than Rényi's jamming limit.

8. Concluding remarks

The free deposition process of rods on a 1D substrate has been studied as a random growth model. For rods with size s (small), hard-rod configurations prevail until the average number of points n is of the order $s^{-1/2}$, which is the average 'time' of some rod overlap. Some of them can still be found when n is of the order ρ/s, $\rho \in (0, 1)$ but the probability of their occurrence is exponentially small there. Packing configurations occur as soon as n is of the order ρ/s, $\rho > 1/2$ but with exponentially small probability, the typical value of their occurrence rather being $(2s)^{-1}L(1/(2s))$. The range of appearance of parking configurations (as packing configurations with no overlap between rods) is thus when n is of the order ρ/s, $\rho \in (1/2, 1)$. Their probability of occurrence is always exponentially small. Covering configurations of the circle are observable as soon as n is of the order ρ/s, $\rho > 1$ but with exponentially small probability, the typical value of their occurrence being $s^{-1}L(1/s)$. With $L(x) := \log(x \log x)$, $x > 1$, this can perhaps be summarized by figure 2.

For physical reasons, one may wish to focus on any of these special types of configurations in the density domain. The problem of the cardinality of the configuration which can typically

$$\underset{2}{\times} - - - - \underset{s^{-1/2}}{\times} - - - - - - \underset{s^{-1}}{\times} - - - - - \underset{(2s)^{-1}L(1/(2s))}{\times} - - - - \underset{s^{-1}L(s^{-1})}{\times} - - \to n$$

overlap density packing covering

$$\textit{Hard Rods} \to\leftarrow - - - - - - - - - - - - - - - \to\leftarrow \textit{Strictly Packing}\to\leftarrow\textit{Covering}$$

$$\textit{Parking}$$

Figure 2. The main types of events and *configurations* encountered as the average number of points n grows with rod length s in a free deposition process. Parking configurations are localized in the density domain. They are exceptional.

be observed has been addressed in the thermodynamic limit. When particularized to parking configurations, our construction is a random selection procedure from the statistical ensemble of all (rare) parking configurations that can arise in a free deposition process, giving equal weight to each of them. This should not be confused with Rényi's model which also selects one of these parking configurations but from sequential physical mechanisms in a deposition process with specific local rules.

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